J. Fluid Mech. (1984), *vol.* 143, *pp.* 243–252
Printed in Great Britain

Particle trajectories in nonlinear capillary waves

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(Received 24 January 1984)

The particle trajectories of nonlinear capillary waves are derived. The properties of the surface and subsurface particles are presented in exact analytic form, up to and including the highest wave. It is found that the orbits of the steeper waves are neither circular nor closed. For the highest wave, a particle moves through a distance $[X]$ equal to 7.99556λ in one orbit, where λ is the wavelength. It moves with an average horizontal drift velocity U equal to $0.88883c$, where c is the phase speed of the wave. In addition, the subsurface particles (at depths nearly three-quarters that of the wavelength) move at speeds up to one-tenth that of surface particles.

1. Introduction

Studies of particle trajectories in steady, steep, symmetric, deep-water gravity waves have revealed interesting results as to the nature of the surface horizontal drift velocity U. This motion of particles in the direction of wave propagation results from the particle orbits not being closed in steep waves. In contrast, for very-small-amplitude waves, the trajectories are circular (elliptic in shallow water) and there is no drift velocity. Two recent papers on the subject have been by Longuet-Higgins (1979) and Srokosz (1981). These studies have used either simple approximations to the steep gravity-wave profile or high-order series expansions.

In the case of capillary waves, an exact solution for the wave profile is available, as derived by Crapper (1957). Using this solution and the general method for calculating particle trajectories given by Longuet-Higgins (1979), we derive exact expressions for the horizontal drift velocity of surface and subsurface particles of a wave of arbitrary height. We also present exact expressions for the coordinates of the trajectories in a frame of reference stationary with respect to great depths. For the highest wave, we find that the surface particles travel nearly eight wavelengths in the course of completing one orbit. They take nearly nine times as long to complete their orbit as do particles at great depths, and their time-averaged drift velocity is almost nine-tenths that of the phase speed of the wave. Strong horizontal motion persists at depths up to three-quarters that of the wavelength.

In §2 we describe the problem of the propagation of nonlinear capillary waves in deep water and introduce the exact solution of Crapper (1957). In §3 we briefly describe the method of Longuet-Higgins (1979) for deriving the trajectories and present exact results for complete orbits of surface and subsurface particles. In §4, we present results for the coordinates of these trajectories in a stationary reference frame, as well as for the drift-velocity ratio U/c as a function of the mean depth of the fluid particles.

2. Capillary waves

We consider steady, symmetric, periodic, nonlinear capillary waves which propagate on the surface of an incompressible, inviscid infinitely deep fluid. The motion in the fluid is taken to be two-dimensional and irrotational and the wave is moving to the right with speed c. We reduce the flow to a steady state by moving in a frame of reference with the waves. We choose Cartesian axes with x measured horizontally to the left and y vertically downwards. For the steady motion, we denote the velocity potential by ϕ and the stream function by ψ. At the free surface, we have $\psi = 0$ and $\psi > 0$ below (increasing y). The trough of one wave corresponds to $\phi = 0$.

This problem was solved exactly for nonlinear waves in a celebrated paper by Crapper (1957). If

$$z = x + \mathrm{i}y \tag{2.1}$$

and

$$\chi = \phi + \mathrm{i}\psi \tag{2.2}$$

and denote the wavenumber by k (so the wavelength $\lambda = 2\pi/k$), then Crapper found

$$z = \frac{\chi}{c} + \frac{4\mathrm{i}}{k}\left\{1 - \frac{1}{1 + A\mathrm{e}^{\mathrm{i}k\chi/c}}\right\} \tag{2.3}$$

where

$$c^2 = \frac{S}{\rho k}\frac{1-A^2}{1+A^2}, \tag{2.4}$$

$$\frac{H}{\lambda} = \frac{4A}{\pi(1-A^2)}. \tag{2.5}$$

Here we denote the surface tension by S and the density by ρ. The crest-to-trough wave height is given by H. Some wave profiles are plotted in figure 1. The highest wave encloses a bubble in its trough, for $H/\lambda = 0.7298$. This solution has the property that any streamline of a wave can be taken as the free surface of a lower wave. Thus, throughout this paper, we shall consider the highest wave and its streamlines only. From (2.5), this means setting $A = 0.4547$ once and for all, although we shall retain the symbol A for brevity.

The set of equations (2.3)–(2.5) ensures the satisfaction of Bernoulli's condition of constant pressure on the free surface, together with uniform flow with speed c to the left at great depths. In addition, Laplace's equation in ϕ is satisfied in the interior. (For full details see Crapper 1957, §2).

3. Method of solution

Longuet-Higgins (1979, §3) sets out a general method for time integration of particle trajectories in steady flows. From his equation (3.3), we find that the time t taken to travel from the point $\phi = \phi_1$ to $\phi = \phi_2$ along the streamline $\psi = \psi_c$ is given by

$$t = \int_{\phi_1}^{\phi_2} \left|\frac{\mathrm{d}z}{\mathrm{d}\chi}\right|^2_{\psi=\psi_c} \mathrm{d}\phi. \tag{3.1}$$

Thus from $z = z(\chi)$ we can find z as a function of t. In a frame of reference stationary with respect to great depths, we subtract the term ct from the horizontal coordinate x to obtain

$$X = x - ct, \quad Y = y. \tag{3.2}$$

The motion in the (X, Y)-plane constitutes the trajectory of a particle.

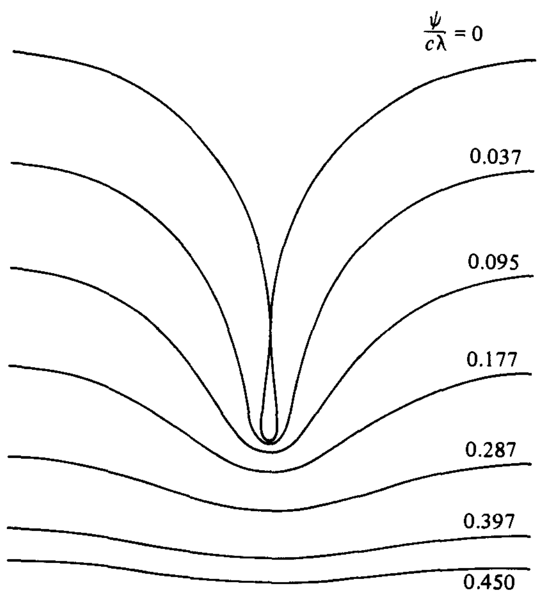

FIGURE 1. Pure capillary-wave streamlines for $\psi/c\lambda = 0$, 0.037, 0.095, 0.177, 0.287, 0.397, 0.450 (after Crapper 1957).

In order to calculate t, we first obtain $\mathrm{d}z/\mathrm{d}\chi$ from (2.3). Thus

$$\frac{\mathrm{d}z}{\mathrm{d}\chi} = \frac{1}{c}\left\{\frac{1-A\mathrm{e}^{\mathrm{i}k\chi/c}}{1+A\mathrm{e}^{\mathrm{i}k\chi/c}}\right\}^2, \tag{3.3}$$

and, from (3.1), we find that the total time T taken to complete one orbit is given by

$$T = \frac{1}{ck}\int_0^{2\pi} \frac{(1+4B^2+B^4-4B(1+B^2)\cos\theta+2B^2\cos 2\theta)}{(1+2B\cos\theta+B^2)^2}\mathrm{d}\theta, \tag{3.4}$$

where we set $\theta = k\phi/c$ and $B = A\exp(-k\psi/c)$. This integral can be calculated explicitly. In fact we have

$$ckT = (1+4B^2+B^4)K_0 - 4B(1+B^2)K_1 + 2B^2K_2, \tag{3.5}$$

where

$$K_n = \int_0^{2\pi} \frac{\cos n\theta}{(1+2B\cos\theta+B^2)^2}\mathrm{d}\theta \quad (n = 0, 1, 2). \tag{3.6}$$

This integral (3.6) can be rewritten as

$$K_n = \mathrm{Re}\oint_{|\zeta|=1} \frac{-\mathrm{i}\zeta^{n+1}}{(1+B\zeta)^2(\zeta+B)^2}\mathrm{d}\zeta, \tag{3.7}$$

where we have set $\zeta = \mathrm{e}^{\mathrm{i}\theta}$. It is a simple matter to calculate the residue at the pole $\zeta = -B$. Note that the pole at $\zeta = -1/B$ lies outside the unit circle since $B < 1$ always, so it does not contribute to the integral.

We find

$$K_n = \frac{2\pi(-B)^n}{(1-B^2)^3}[(n+1)-(n-1)B^2], \tag{3.8}$$

and so from (3.5) and (3.8) we have

$$\frac{cT}{\lambda} = \frac{1+13B^2+19B^4-B^6}{(1-B^2)^3}. \tag{3.9}$$

Equation (3.9) is an exact result for the total time taken to complete one orbit along the streamline $\psi = \psi_c$. Thus for the free surface of the highest wave, where $\psi_c = 0$, we find

$$\frac{cT}{\lambda} = 8.9956, \tag{3.10}$$

or since

$$c = 2.0323\left(\frac{S}{\lambda\rho}\right)^{\frac{1}{2}} \tag{3.11}$$

from (2.4), we have

$$T = 4.4262\left(\frac{\lambda^3\rho}{S}\right)^{\frac{1}{2}}. \tag{3.12}$$

Thus a particle on the free surface of the highest pure capillary wave takes nearly nine times longer to complete its orbit than does a particle at great depths. For pure gravity waves, Longuet-Higgins (1979) has shown that $cT/\lambda = 1.377$.

In time T the particle has advanced a distance $cT-\lambda$, and so its average speed U is given by

$$U = \frac{cT-\lambda}{T} = c\left(1-\frac{\lambda}{cT}\right).$$

From (3.9) we find

$$\frac{U}{c} = \frac{16B^2(1+B^2)}{1+13B^2+19B^4-B^6}, \tag{3.13}$$

and for the highest wave we have

$$\frac{U}{c} = 0.8888. \tag{3.14}$$

This is to be compared with the result for the highest gravity wave, where $U/c = 0.274$.

Also in time T, the particle moves through a certain distance $[X]$, conveniently expressed as a proportion of the main wavelength:

$$\frac{[X]}{\lambda} = \frac{cT-\lambda}{\lambda} = \frac{cT}{\lambda}-1.$$

Thus from (3.9) we have

$$\frac{[X]}{\lambda} = \frac{16B^2(1+B^2)}{(1-B^2)^3}, \tag{3.15}$$

and for the highest wave we find

$$\frac{[X]}{\lambda} = 7.9956. \tag{3.16}$$

For the highest gravity wave the corresponding result is $[X]/\lambda = 0.337$.

In the limit of small amplitude (that is retaining only terms quadratic in the wave steepness), the results given by (3.9), (3.13) and (3.15) agree with the classical solution of Stokes (see Lamb 1932, chap. 9).

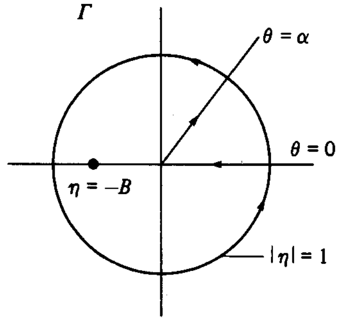

FIGURE 2. The contour Γ used to evaluate the integral in (4.1).

4. Surface and subsurface particle trajectories

In order to calculate the exact position of each particle using (3.2), we have to know the time it takes to reach that position, starting from (say) the trough. In other words, we must evaluate (3.1) from $\phi = 0$ to $\phi = \alpha c/k$, where $0 \leqslant \alpha \leqslant 2\pi$. To do this we consider the integral

$$K_{p-1}(\alpha) = \mathrm{Re} \oint_{\Gamma} \frac{-\mathrm{i}\eta^p}{(1+B\eta)^2(\eta+B)^2}\,\mathrm{d}\eta, \tag{4.1}$$

where $\eta = r\mathrm{e}^{\mathrm{i}\theta}$, the contour Γ is given in figure 2, and p is a positive integer. By considering the three segments of the contour $\{\theta = \alpha; r = 0 \rightarrow 1\}$, $\{\theta = \alpha \rightarrow 2\pi;\ r = 1\}$ and $\{\theta = 0;\ r = 1 \rightarrow 0\}$, we find

$$\oint_{\Gamma} \frac{-\mathrm{i}\eta^p}{(1+B\eta)^2(\eta+B)^2}\,\mathrm{d}\eta = \int_{\alpha}^{2\pi} \frac{\mathrm{e}^{\mathrm{i}(p-1)\theta}}{(1+B\mathrm{e}^{\mathrm{i}\theta})^2(1+B\mathrm{e}^{-\mathrm{i}\theta})^2}\,\mathrm{d}\theta$$
$$+\mathrm{i}\int_0^1 \frac{r^p}{(1+Br)^2(B+r)^2}\,\mathrm{d}r - \mathrm{i}\int_0^1 \frac{r^p\mathrm{e}^{\mathrm{i}(p+1)\alpha}}{(1+Br\mathrm{e}^{\mathrm{i}\alpha})^2(B+r\mathrm{e}^{\mathrm{i}\alpha})^2}\,\mathrm{d}r. \tag{4.2}$$

We note that the value of the contour integral on the left-hand side of (4.2) is unchanged if we deform Γ into the contour $r = 1$, that is, the unit circle. In fact,

$$\oint_{|\eta|=1} \frac{-\mathrm{i}\eta^p}{(1+B\eta)^2(\eta+B)^2}\,\mathrm{d}\eta = \int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}(p-1)\theta}}{(1+B\mathrm{e}^{\mathrm{i}\theta})^2(1+B\mathrm{e}^{-\mathrm{i}\theta})^2}\,\mathrm{d}\theta. \tag{4.3}$$

Thus from (4.2) and (4.3) we find

$$\int_0^{\alpha} \frac{\cos(p-1)\theta}{(1+B\mathrm{e}^{\mathrm{i}\theta})^2(1+B\mathrm{e}^{-\mathrm{i}\theta})^2}\,\mathrm{d}\theta = \mathrm{Re}\{-\mathrm{i}\mathrm{e}^{\mathrm{i}(p-1)\alpha} I_p\}, \tag{4.4}$$

where

$$I_p = \int_0^1 \frac{r^p}{(1+Cr)^2(C^*+r)^2}\,\mathrm{d}r \tag{4.5}$$

and

$$C = B\mathrm{e}^{\mathrm{i}\alpha}. \tag{4.6}$$

By analogy with (3.4) we require $p = 1, 2, 3$ only.

We find

$$I_1 = \frac{1+|C|^2}{(1-|C|^2)^3}\log\frac{(1+C^*)}{C^*(1+C)} - \frac{1+2C+|C|^2}{(1-|C|^2)^2(1+C)(1+C^*)} \tag{4.7}$$

and

$$I_2 = \frac{2C^*}{(1-|C|^2)^3}\log\frac{(1+C^*)}{C^*(1+C)} + \frac{1+2C^*+|C|^2}{(1-|C|^2)^2(1+C)(1+C^*)}, \tag{4.8}$$

and that

$$I_3 = \frac{1}{C^2(1-|C|^2)^3}\left\{(1-3|C|^2)\log(1+C) + |C|^4(3-|C|^2)\log\frac{1+C^*}{C^*}\right\} - \frac{1+C^*(1+|C|^2)+|C|^4}{C(1-|C|^2)^2(1+C)(1+C^*)}. \tag{4.9}$$

Then, by substituting (4.7)–(4.9) into (4.4) and using (4.6), we obtain

$$K_0(\alpha) = \frac{1+B^2}{(1-B^2)^3}(\alpha-\mu) - \frac{2B\sin\alpha}{(1-B^2)^2(1+B^2+2B\cos\alpha)}, \tag{4.10}$$

$$K_1(\alpha) = \frac{1}{(1-B^2)^3}\left\{2B(\mu-\alpha) + \frac{(1-B^4)\sin\alpha}{1+B^2+2B\cos\alpha}\right\}, \tag{4.11}$$

and that

$$K_2(\alpha) = \frac{1}{B^2(1-B^2)^3}\{B^4(3-B^2)\alpha + \tfrac{1}{2}\mu(1+B^2)(1-4B^2+B^4)\} - \frac{(1+B^4)\sin\alpha}{B(1-B^2)^2(1+B^2+2B\cos\alpha)}, \tag{4.12}$$

where

$$\tan\tfrac{1}{2}\mu = \frac{B\sin\alpha}{1+B\cos\alpha}. \tag{4.13}$$

The parameter μ occurs naturally when taking the imaginary part of the log terms in (4.7)–(4.9).

The time $t(\alpha)$ to reach the point $\phi = \alpha c/k$ on the profile from the trough is given, by analogy with (3.4), by

$$ckt(\alpha) = (1+4B^2+B^4)K_0(\alpha) - 4B(1+B^2)K_1(\alpha) + 2B^2K_2(\alpha). \tag{4.14}$$

Upon substituting (4.10)–(4.12) into (4.14) we eventually find

$$ckt(\alpha) = \frac{\alpha(1+13B^2+19B^4-B^6)}{(1-B^2)^3} - \frac{16\mu B^2(1+B^2)}{(1-B^2)^3} - \frac{8B(1+B^2)^2\sin\alpha}{(1-B^2)^2(1+B^2+2B\cos\alpha)}. \tag{4.15}$$

When $\alpha = 2\pi$ we have $\sin\alpha = 0$, $\mu = 0$ and $t(2\pi) = T$, so we recover (3.9), as expected.

In order to calculate the orbit of the particle, we must evaluate X and Y from (3.2). From (2.3) we find

$$kx = \alpha - \frac{4B\sin\alpha}{1+B^2+2B\cos\alpha}, \tag{4.16}$$

$$ky = \frac{k\psi}{c} + \frac{4B(B+\cos\alpha)}{(1+B^2+2B\cos\alpha)}, \tag{4.17}$$

and so, using (4.15) and (3.2), we get

$$kX = \frac{16B^2(1+B^2)}{(1-B^2)^3}(\mu-\alpha) + \frac{4B(1+6B^2+B^4)\sin\alpha}{1+B^2+2B\cos\alpha}, \tag{4.18}$$

$$kY = \frac{k\psi}{c} + \frac{4B(B+\cos\alpha)}{1+B^2+2B\cos\alpha}, \tag{4.19}$$

where $B = 0.4547\exp(-k\psi/c)$.

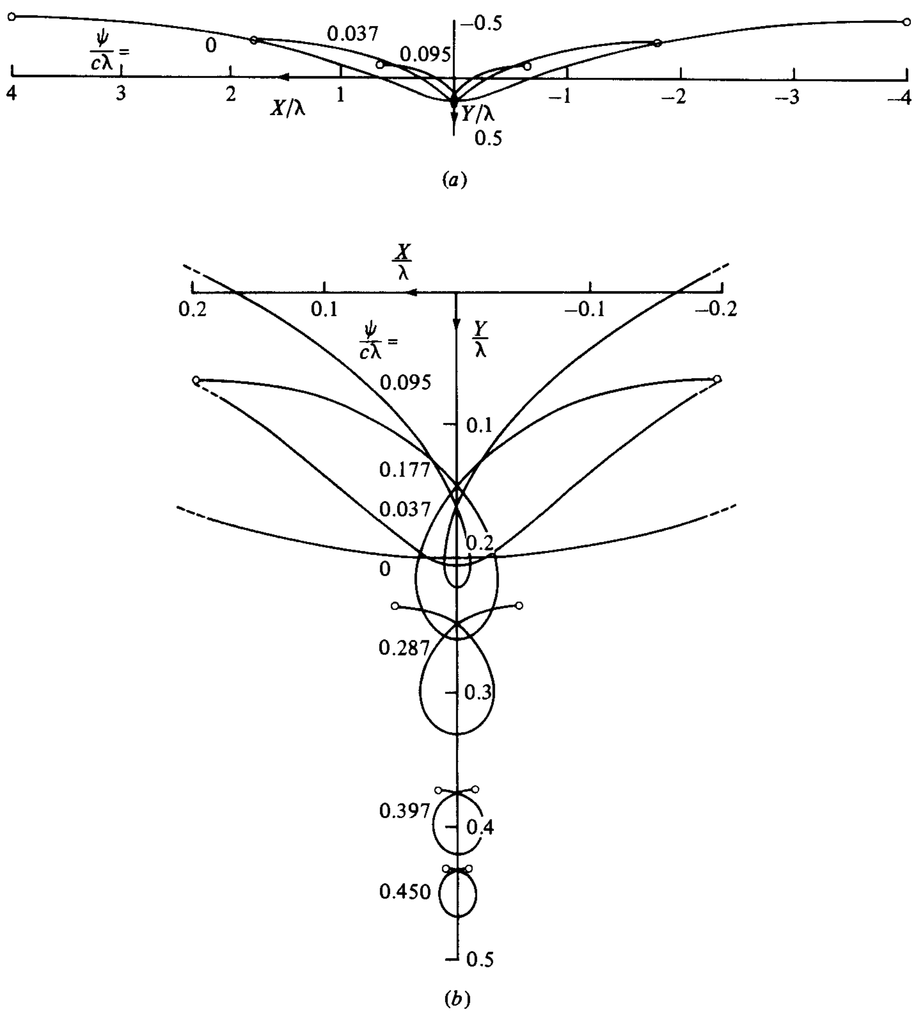

FIGURE 3. (a) Particle trajectories for capillary waves along streamlines $\psi/c\lambda = 0, 0.037$, and 0.095; (b) Full trajectories for $\psi/c\lambda = 0.177$, 0.287, 0.397, 0.450, together with part trajectories from figure 3(a).

Equations (4.18) and (4.19) are exact, nonlinear expressions for the particle trajectories of pure capillary waves. In the limit of great depths below the surface (or equivalently small-amplitude waves), $B \ll 1$ and we find $kX \approx 4B\sin\alpha$, $kY - k\psi/c \approx 4B\cos\alpha$; that is,

$$(kX)^2 + \left(kY - \frac{k\psi}{c}\right)^2 = 16B^2. \tag{4.20}$$

This corresponds to circular paths of radius $4B/k$, centre $(0, \psi/c)$.

The particle trajectories of waves given in figure 1 are drawn in figures 3(a, b). This division is necessary for reasons of scale. In figure 3(a) we have illustrated complete trajectories for $\psi/c\lambda = 0$, 0.037 and 0.095. In figure 3(b) we have trajectories for $\psi/c\lambda = 0.177, 0.287, 0.397$ and 0.450, as well as the relevant parts of the trajectories of figure 3(a). We can clearly see that as $\psi/c\lambda$ increases, the trajectories become less open and more circular, tending to the form given in (4.20). The most striking feature of figure 3 is the enormous distance through which the surface particles of the high

$\frac{\psi}{c\lambda}$	$\frac{H}{\lambda}$	$\frac{[X]}{\lambda}$	$\frac{cT}{\lambda}$	$\frac{U}{c}$
0	0.7298	7.9956	8.9956	0.8888
0.037	0.5274	3.5632	4.5632	0.7808
0.095	0.3400	1.2934	2.2934	0.5640
0.177	0.1947	0.3914	1.3914	0.2812
0.200	0.1676	0.2866	1.2866	0.2228
0.250	0.1214	0.1481	1.1481	0.1290
0.287	0.0959	0.0918	1.0918	0.0841
0.300	0.0883	0.0777	1.0777	0.0721
0.350	0.0643	0.0411	1.0411	0.0395
0.397	0.0478	0.0227	1.0227	0.0222
0.450	0.0343	0.0116	1.0116	0.0115
0.500	0.0250	0.0062	1.0062	0.0062

TABLE 1. Particle parameters for complete orbits of particles in capillary waves

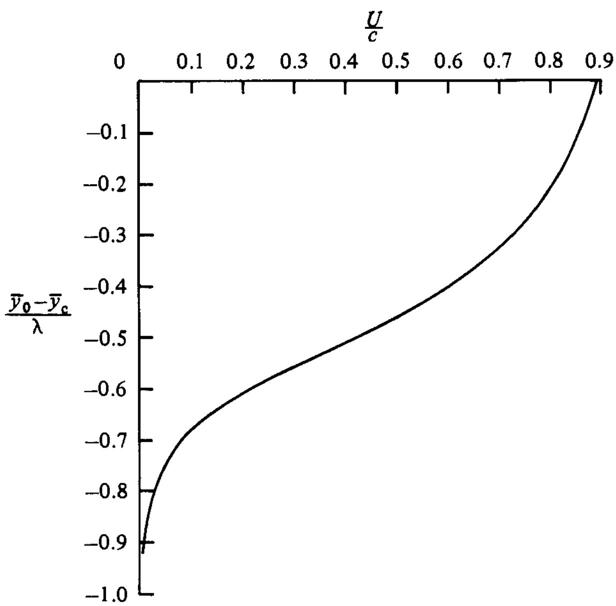

FIGURE 4. Drift velocity ratio U/c as a function of the mean displacement of fluid particles $(\bar{y}_0-\bar{y}_c)/\lambda$.

waves travel. Their orbits are not closed at all. This phenomenon persists until $\psi/c\lambda = 0.1$ approximately, where the particle still advances over one wavelength in each orbit. When $\psi/c\lambda = 0.06998$ ($B = 0.29291$) there is a cusp in the orbit.

At greater depths the particle orbits involve a circular form of motion. This has a dramatic effect on the distance advanced in one orbit, but not on the time taken. Details of all the orbits in figure 3 are given in table 1. There we see that although the surface particles take nearly nine times as long to complete one orbit as do the particles at $\psi/c\lambda = 0.450$, they travel nearly seven hundred times as far. Put another way, in the time it takes a surface particle of the highest wave to complete one orbit,

$\frac{X}{\lambda}$	$\frac{Y}{\lambda}$	$\left(\frac{S}{\lambda^3 \rho}\right)^{\frac{1}{2}} t$
0	0.1990	0
−0.0632	0.1953	0.0351
−0.1331	0.1833	0.0724
−0.2193	0.1599	0.1156
−0.3396	0.1185	0.1727
−0.5334	0.0455	0.2638
−0.9001	−0.0844	0.4450
−1.3557	−0.2177	0.6862
−2.1264	−0.3807	1.1176
−3.3026	−0.5108	1.8026
−3.9978	−0.5308	2.2131

TABLE 2. Coordinates and arrival times from the trough of particles on the surface of the highest capillary wave

it has advanced more than 77.5 times the distance travelled by a particle at $\psi/c\lambda = 0.450$ in the same time.

In table 1 we have also given results for the drift velocity U as a function of the phase speed c of the wave. It turns out to be more convenient to plot U/c against the mean displacement of the streamline from the surface. Thus from (4.19) we calculate $\overline{y/\lambda}$, where

$$\bar{y} \equiv \frac{1}{\lambda}\int_0^\lambda y\,\mathrm{d}x. \tag{4.21}$$

Thus, using (4.17), (4.21) and results from §3 of Hogan (1979), we find along $\psi = \psi_c$,

$$\left(\frac{\overline{y_c}}{\lambda}\right) = \frac{\psi_c}{c\lambda} - \frac{4B^2}{\pi(1-B^2)^2}, \tag{4.22}$$

and so the mean level of the free surface of the highest wave $\psi_c = 0$, $B = 0.4547$ is $\overline{y_0/\lambda} = -0.4183$. In figure 4 we have plotted U/c against $(\bar{y}_0 - \bar{y}_c)/\lambda$. We note that a significant amount of fluid is being transported forward. In fact we have to go to a depth of $\frac{3}{4}\lambda$ to reduce U/c to one-tenth of its value at the free surface. The classical result for this case can be written as

$$\frac{U}{c} = 16A^2 \exp\left(-2k(\bar{y}_c - \bar{y}_0)\right), \tag{4.23}$$

where $\bar{y}_0 = 0$ and $\bar{y}_c = \psi_c/c$. At great depths, that is, large value of ψ_c/c, we find that this result is in very close agreement with the exponential tail of figure 4. At the surface, however, (4.23) gives $U/c = 3.3076$. This is a considerable overprediction of the exact result, $U/c = 0.8888$, given in (3.14).

Finally, in table 2 we present results for the coordinates of particle trajectories for the surface streamline of the highest capillary wave. It is clear that the particle spends very little time in the neighbourhood of the trough.

5. Discussion

The remarkable results contained in §§3 and 4 show conclusively that the orbits in steep capillary waves are neither circular nor closed. It must be emphasized just how much of the bulk of fluid moves forward during the passage of a steep capillary wave. In addition the particles spend most of their time near the wave crests. These results contrast strongly with those derived by Longuet-Higgins (1979) and Srokosz (1981) for steep gravity waves.

The effects of finite depth can be included using the work of Kinnersley (1976), who derived exact solutions in a similar manner to Crapper (1957). Gravity can also be included, albeit numerically, using the high-order solutions of Hogan (1980, 1981).

Other effects have been neglected, including viscosity and surface-tension gradients. In this latter case, another type of motion is possible, namely longitudinal waves (Lucassen 1968). This occurs because the surface-tension gradient can support a tangential stress difference across the interface. Even for infinitesimal wave amplitudes, the particle trajectories are known to be non-circular (but still closed), with the exact form being a functon of the extensibility, or dilational modulus, of the surface (Lucassen-Reynders & Lucassen 1969, figure 4). Even in the absence of surface-tension gradients, viscosity will affect the behaviour at the surface, owing to the presence of a boundary layer of thickness $\sim (\nu/\sigma)^{\frac{1}{2}}$, where ν is the kinematic viscosity and σ is the frequency of the waves. It is well known that for gravity waves, viscosity increases the drift velocity, with the velocity gradient being double that of the inviscid case (Longuet-Higgins 1953, 1960).

The author gratefully acknowledges support from King's College, Cambridge, in the form of a Junior Research Fellowship. This work was initiated by a remark of Dr J. C. Scott concerning some of his unpublished experimental results.

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